

NOTES ON MATRIX THEORY--IV (An Inequality due to Bergström)

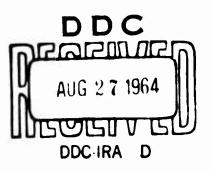
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## SUMMARY

Two proofs are presented of an inequality due to Bergström.

# NOTES ON MATRIX THEORY--IV (An Inequality due to Bergstrom)

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### 51. Introduction.

In a recent note, Bergstrom proved the following interesting inequality:

"Let A and B be positive definite matrices and let  $A_{ii}$ ,  $B_{ii}$  denote the sub-matrices obtained by deleting the  $i^{th}$  row and column. Then

$$\frac{|A+B|}{|A_{1i}+B_{1i}|} \geq \frac{|A|}{|A_{1i}|} + \frac{|B|}{|B_{1i}|}.$$

where | | represents the determinant."

Bergstrom's proof is essentially a verification. We present two proofs below, the second of which lays bare the origin of the result.

#### §2. First Proof:

The first proof is an immediate consequence of the result:

#### Lemma 1: If A is positive definite, then

(1) 
$$\phi(\Lambda) = \frac{|\Lambda|}{|\Lambda_{ii}|} = \min_{X} \sum_{i,j=1}^{N} a_{ij} x_i x_j,$$

where x is constrained by

H. Bergstrom, "A Triangle Inequality for Matrices," <u>Den 11te skandinaviske Matematikerkongress, Trondheim, 1949</u>; Oslo, 1952; pp. 264-267.

$$(2) x_i = 1.$$

from this it is clear that  $\phi(A+B) \ge \phi(A) + \phi(B)$ .

We shall not present the proof, which is easily obtained by the use or a Lagrange multiplier, since Lemma 1 is a special case of the more general result established in the next section.

#### §3. Second Proof:

We begin by establishing

Lemma 2: If A is positive definite, then

(1) 
$$(x, \Lambda x) (y, \Lambda^{-1} y) \ge (x, y)^2$$
,

for all x and y.

Here (x,y) denotes the inner product of x and y and  $(x,\Lambda x)$  the quadratic form  $\sum_{i,j}^{-1} a_{i,j} x_i x_j$ .  $A^{-1}$  is the inverse of A.

Proof of Lemma 2: Reduce A to diagonal form by an orthogonal matrix T, i.e., T'AT = L,  $T' = T^{-1}$ . Let x = Tu, y = Tv. Then (1) becomes

(2) 
$$\left(\sum_{i=1}^{N} x_{i} u_{i}^{2}\right) \left(\sum_{i=1}^{N} v_{i}^{2} / x_{i}\right) = \sum_{i=1}^{N} u_{i} v_{i}$$

which is the Cauchy-Schwarz inequality.

Since the inequality becomes an equality for suitable choice of  $\mathbf{x}_{\bullet}$  we have

(3) 
$$\lim_{x} \frac{(x, Ax)}{(x, y)^2} = \frac{1}{(y, A^{-1}y)} = \psi(A).$$

From this it is immediate that

(4) 
$$\Psi(A+B) \geq \Psi(A) + \Psi(B).$$

The case  $y_i=1$ ,  $y_j=0$ ,  $j\neq i$  yields Bergstrom's result.

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